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Secondary instability of a gas-fluidized bed

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It is known that stationary fluid with density that varies sinusoidally with small amplitude and wavenumber κ in the vertical direction is unstable to disturbances that are sinusoidal in a horizontal direction with wavenumber α. Small values of α/κ are the most unstable in the sense that a neutral disturbance exists at sufficiently small α/κ however small the Rayleigh number may be. The non-uniformity of density in the undisturbed state may be regarded as being a consequence of non-uniformity of concentration of extremely small solid particles in fluid. This paper is concerned with the corresponding instability of such a non-uniform dispersion when the particle size is not so small that the fall speed relative to the fluid is negligible. In the undisturbed state, which is an outcome of the well-known primary instability of a uniform fluidized bed with particle volume fraction ϕ_0, the sinusoidal distribution of concentration propagates vertically, and in the steady state relative to this kinematic wave particles fall with speed $V\ (= |\phi\, \mathrm{d}U/\mathrm{d}\phi|_{\phi_0})$, where $U(\phi)$ is the mean speed of fall of particles, relative to zero-volume-flux axes, in a uniform dispersion with volume fraction ϕ. This particle convection with speed V transports particle volume and momentum and tends to even out variations of a disturbance in the vertical direction and thereby to suppress a disturbance, especially one with small α/κ. Analysis of the behaviour of a disturbance is based on the equation of motion of the mixture of particles and fluid and an assumption that the disturbance velocities of the particles and the fluid are equal (as is suggested by the relatively small relaxation time of particles). The method of solution used in the associated pure-fluid problem is also applicable here, and values of the Rayleigh number as a function of α/κ for a neutral disturbance and a given value of the new non-dimensional parameter involving V are found. Particle convection with only modest values of V stabilizes all disturbances for which $\alpha/\kappa < 1$ and increases significantly the Rayleigh number for a neutral disturbance when $\alpha/\kappa > 1$. It appears that under practical conditions disturbances with α/κ above unity are unstable, although ignorance of the values of parameters characterizing a fluidized bed hinders quantitative conclusions.

1. Introduction

It is generally accepted that under certain conditions a statistically uniform fluidized bed of solid particles is unstable to small disturbances having the form of plane sinusoidal waves with horizontal wave fronts. The instability is caused essentially by inertia forces on the particles, as was first made clear by Jackson (1963). The physical factor or process that tends to suppress the disturbance and that yields a criterion for instability is less evident. The theoretical evidence points to a condition for instability of the form

$$\text{particle Froude number } \frac{U_0^2}{ag} > \text{critical value of order unity,}$$

where a is the radius of the (spherical) particles and U_0 is the speed of fall of an isolated particle. If one is willing to assume a specific mechanism for suppression of the disturbance, in particular a particle stress related to the diffusive transport of particles down a concentration gradient (Batchelor 1988), the critical value of the Froude number may be found to a rough approximation.

A nearly neutral wavy disturbance propagates vertically upwards with the kinematic wave speed

$$V = \left| \phi \frac{\mathrm{d}U}{\mathrm{d}\phi} \right|_{\phi=\phi_0} \tag{1.1}$$

relative to the particles, where $U(\phi)$ is the mean speed of fall of particles, relative to zero-volume-flux axes, in a homogeneous dispersion with particle volume fraction ϕ. At Froude numbers only a little above the critical value, the range of values of the vertical wavenumber (κ) for which the growth rate of the disturbance is positive is given by $0 < \kappa < \kappa_n$, where κ_n is the wavenumber of a neutral disturbance; and the wavenumber for which the growth rate is a maximum is of the same order of magnitude as κ_n.

Vertically propagating concentration waves which form spontaneously have been observed in liquid-fluidized beds, although only a few photographs have been published (see El-Kaissy & Homsy 1976; Didwania & Homsy 1981). Even rarer are pictures of growing waves in gas-fluidized beds, possibly because in gas-fluidized beds, which are known to be more unstable, disturbances grow very rapidly and quickly change form. Bubbles of relatively clear fluid are a prominent feature of both gas-fluidized beds used in industrial plants and those employed in the laboratory, and it is a common speculation (Didwania & Homsy 1981; Batchelor 1991) that they are a direct consequence of nonlinear processes in a growing concentration wave.

In the latter of these two references I suggested that the transition from an (unstable) uniform fluidized bed to a bed containing steadily rising bubbles may be thought of as taking place in four stages:

(i) plane concentration waves in a uniform bed grow exponentially in amplitude;

(ii) the vertical gradients of mean density of the mixture of particles and fluid resulting from (i) become so large that a secondary overturning instability develops;

(iii) nonlinear processes then lead to the formation of compact regions of smaller-than-average concentration which rise and develop an internal circulation of fluid which expels the remaining particles by centrifugal action;

(iv) the rising bubble of almost clear fluid develops a steady shape and motion.

Stage (i) has already been investigated in the references cited; the purpose of the present note is to examine stage (ii); stage (iii) is the subject of a separate paper (Batchelor & Nitsche 1994); and stage (iv) is a natural development which could perhaps be analysed using the available models of bubble motion in fluidized beds (see Davidson, Harrison & Guedes de Carvalho 1977).

This, then, is the background to the following investigation of the secondary overturning instability of a fluidized bed through which a wave with a sinusoidal variation of mean density in the vertical direction is propagating with speed V. The fluidizing fluid is taken to be a gas, because bubble formation in this case is more common and the equations governing the two-phase flow are simpler when the fluid density is effectively zero. It will be assumed, for obvious reasons of mathematical convenience, that the amplitude of the primary concentration wave is small and steady, and then if we chose axes moving with the wave the sinusoidal variation of mean density generated by the primary instability is stationary. It will also be assumed that the amplitude of the primary concentration wave is independent of height, even though in a practical situation in which the source of the disturbance is located at the base of the bed an unstable wave is like to grow exponentially with vertical distance rather than with time. This is the 'undisturbed' state whose stability will be investigated by superimposing a small disturbance which is sinusoidal with wavenumber α in an arbitrary horizontal direction. Note that relative to these axes the mean speed of fall of the particles is approximately constant and equal to V everywhere owing to the smallness of the variation of particle concentration in the undisturbed state.

In addition to its relevance to the intriguing phenomenon of bubble formation in fluidized beds, the present investigation may have some interest as an addition to the rather small number of cases of two-phase flow which can be analysed without the need for arbitrary hypotheses concerning the equations governing the motion.

It may be noted that another and less simple type of secondary instability of liquid-fluidized beds in which the transverse structure develops as a consequence of a weakly nonlinear resonant sideband instability of the primary concentration wave has been proposed by Didwania & Homsy (1982) and investigated quantitatively on the assumption that the particle phase behaves like a second fluid.

2. Results concerning the instability of stationary fluid with a sinusoidal variation of density

In the limit in which $a \rightarrow 0$ without change of the particle density or the local particle volume fraction ϕ, there is no 'slip' of the particles relative to the fluid and the dispersion behaves dynamically like a fluid of non-uniform density. The stability of a stationary continuous fluid with a sinusoidal distribution of density in the vertical direction thus represents a limiting case of our two-phase flow problem, and is an interesting stability problem in its own right. This fluid-stability problem has been addressed in two recent papers, one (Batchelor & Nitsche 1991, referred to herein as BN1) in which the fluid is assumed to be unbounded, and the second (Batchelor & Nitsche 1993, referred to herein as BN2) in which the effect of a vertical cylindrical boundary is allowed for. It will be useful to recapitulate here the results of this investigation, because our approach to the related two-phase flow problem is essentially to modify appropriately the solution of the continuous-fluid flow problem.

The linearized equations governing a disturbance to stationary fluid with density $\rho_1(z)$ solved in BN1 are

$$\nabla \cdot \boldsymbol{u} = 0, \tag{2.1}$$

$$\rho_0 \frac{\partial \boldsymbol{u}}{\partial t} = \rho' \boldsymbol{g} - \nabla p' + \mu \nabla^2 \boldsymbol{u}, \tag{2.2}$$

$$\frac{\partial \rho'}{\partial t} + w \frac{\mathrm{d}\rho_1}{\mathrm{d}z} = D \nabla^2 \rho', \tag{2.3}$$

where $\boldsymbol{u}$ (components u, v, w) is the disturbance velocity and a prime to ρ or p indicates a disturbance quantity. The fluid density in the undisturbed state is

$$\rho_0 + \rho_1(z) = \rho_0(1 + A \sin \kappa z), \tag{2.4}$$

where $|A| \ll 1$, allowing use of the Boussinesq approximation. D is the diffusivity of whatever physical property (e.g. solute concentration) is responsible for the variability of the fluid density. The dependent variables u, v, ρ', p' may be eliminated from (2.1), (2.2) and (2.3), giving

$$\left(\frac{\partial}{\partial t}-D\nabla^2\right)\left(\frac{\partial}{\partial t}-\nu\nabla^2\right)\nabla^2 w = \frac{g}{\rho_0}\frac{\mathrm{d}\rho_1}{\mathrm{d}z}\nabla_h^2 w, \tag{2.5}$$

where $\nu = \mu/\rho_0$ and ∇_h^2 denotes the Laplacian in the horizontal plane.

The assumed form of the disturbance is

$$w = \mathrm{e}^{\gamma t} W(z)\cos\alpha x, \tag{2.6}$$

whence (2.5) becomes

$$\left(\frac{\gamma}{D}+\alpha^2-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\left(\frac{\gamma}{\nu}+\alpha^2-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)\left(\alpha^2-\frac{\mathrm{d}^2}{\mathrm{d}z^2}\right)W = \frac{g\alpha^2}{D\nu\rho_0}\frac{\mathrm{d}\rho_1}{\mathrm{d}z}W, \tag{2.7}$$

where α is the wavenumber of the disturbance in an arbitrary horizontal direction. It will be seen later that the most unstable disturbances in the corresponding two-phase flow problem are those for which α/κ is $O(1)$ or larger. If now a vertical circular cylindrical rigid boundary with radius large compared with κ^{-1} (a condition that is normally satisfied) is present, the boundary may be expected to have little effect on these most unstable disturbances; and we shall therefore assume for simplicity that no boundary is present, either in the non-uniform fluid problem being described here or in the two-phase problem considered later. The only dimensionless parameters appearing in the fluid-instability problem are then

$$\frac{\alpha}{\kappa},\quad \frac{\nu}{D},\quad R\left(=\frac{gA}{D\nu\kappa^3}\right). \tag{2.8}$$

A solution of (2.7) such that $W(z)$ is a periodic function of z with period $2\pi/\kappa$ may be written as

$$W(z) = \sum_{n=1}^{\infty} F_n \sin n\kappa z + \sum_{n=0}^{\infty} G_n \cos n\kappa z. \tag{2.9}$$

Substitution of (2.9) in (2.7), conversion of all terms to Fourier series, and the equation of coefficients yields two three-term recurrence relations, one for the coefficients F_n and one for the G_n. It was found possible in BN1 to solve these recurrence relations with high accuracy by truncating the two Fourier series in (2.9) after only a few terms. Many modes of disturbance exist, and the one for which the Rayleigh number R has the smallest value at given γ and α/κ is found to be an even function of z for which

$$\left(\frac{\gamma}{D}+\alpha^2\right)\left(\frac{\gamma}{\nu}+\alpha^2\right)\left(\frac{\gamma}{D}+\kappa^2+\alpha^2\right)\left(\frac{\gamma}{\nu}+\kappa^2+\alpha^2\right) = \frac{\frac{1}{2}\alpha^2\kappa^8}{\kappa^2+\alpha^2}R^2. \tag{2.10}$$

(Sub-harmonic disturbances which are periodic in z with a period equal to an integral multiple of $2\pi/\kappa$ also exist, but were found to be less unstable than the synchronous mode described above.)

The relation (2.10) shows the remarkable result that for any $R > 0$ there is a range of values of α/κ bounded above for which $\gamma > 0$. For an even neutral disturbance ($\gamma = 0$) (2.10) reduces to

$$R = \frac{\surd 2\alpha(\kappa^2+\alpha^2)^{\frac{3}{2}}}{\kappa^4}, \tag{2.11}$$

which is shown graphically as the curve labelled $J = 0$ in the later figure 1. The asymptotic form of (2.11) is

$$R \sim \sqrt{2\alpha/\kappa} \quad \text{as} \quad \alpha/\kappa \to 0,$$

showing that unstable disturbances exist for small values of α/κ, however small the Rayleigh number, although the growth rate goes to zero with R. A sinusoidal density distribution in continuous fluid is evidently always unstable. Of course, the presence of a circular cylindrical boundary with radius b would affect the behaviour of disturbances with values of α comparable with b^{-1} or smaller (BN2); but we are ignoring boundary effects in this resumé of results for pure-fluid instability.

It will help, in our later consideration of the corresponding problem for a dispersion of particles, to have a picture of the way in which disturbances with large horizontal wavelength efficiently convert potential energy to kinetic energy. Figure 6 in BN1 shows the sense of vertical and horizontal components of velocity in the neutral even-mode disturbance represented by (2.11) when $\alpha/\kappa = 0.1$, and indicates a tilting of the layers of smaller and larger density accompanied by a sliding of these layers alternately up and down and accumulation of lighter and heavier fluid in alternate vertical columns, thereby reinforcing the initial tilting.

3. Equations governing the instability of a particle dispersion with a steady sinusoidal variation of concentration

We try now to make a similar calculation of instability of a two-phase medium consisting of sedimenting rigid particles of density ρ_p dispersed in a fluid of much smaller density. The mean density of this two-phase medium in the undisturbed state is assumed to be

$$\rho_p\{\phi_0 + \phi_1(z)\} = \rho_p\,\phi_0(1 + A\sin\kappa z), \tag{3.1}$$

and if we write $\rho_p\,\phi_0 = \rho_0$ (3.1) is identical with (2.4). The distribution of mean density in the undisturbed state is steady relative to axes moving with the concentration wave, as explained in §1, and particles are falling through the wave with approximately uniform mean speed equal to V (see (1.1)). If the particle radius is very small this 'slip' velocity is effectively zero and the mixture behaves like a continuous fluid. We need to consider the consequences of this slip velocity not being zero.

For this purpose we introduce the equation of motion of the mixture (a procedure that has not often been exploited in past work on two-phase flow). The mixture is a mobile continuum, and if we assume for convenience that the fluctuations in particle velocity are isotropic the equation of motion of the mixture is of Navier–Stokes form. Let $\boldsymbol{u}_p$ denote the mean disturbance velocity of the particles and $\boldsymbol{u}_m$ that of the mixture ($\boldsymbol{u}_m$ being defined as the ensemble average of the velocity at a point regardless of whether the point lies instantaneously in a particle or in the fluid). The two constituents of the mixture are volume preserving, so

$$\nabla\cdot\boldsymbol{u}_m = 0. \tag{3.2}$$

The components of particle velocity in the undisturbed state are $(0, 0, -V)$. The particle mass per unit volume multiplied by the mean acceleration of the particles in the disturbed state is thus

$$\rho_0\left(\frac{\partial \boldsymbol{u}_p}{\partial t} - V\frac{\partial \boldsymbol{u}_p}{\partial z}\right)$$

correct to first order in disturbance quantities. There is a similar contribution to the inertia force on the mixture from the (gaseous) fluid, but it is negligible. The linearized equation of motion of the mixture then has the form

$$\rho_0\left(\frac{\partial \boldsymbol{u}_p}{\partial t}-V\frac{\partial \boldsymbol{u}_p}{\partial z}\right)=\rho'\boldsymbol{g}-\boldsymbol{\nabla}p'+\mu_m\nabla^2\boldsymbol{u}_m, \tag{3.3}$$

where $\rho'(=\rho_p\phi')$ and p' are the density and pressure perturbations for the mixture. By the term 'pressure' we mean simply $(-\frac{1}{3})$ times the ensemble average of the trace of the stress tensor at a point in the mixture regardless of whether the point lies in a particle or in the fluid.

The coefficient μ_m is the effective viscosity of the mixture. Note that μ_m includes the transport of momentum by particles with random velocities and so may be much larger in magnitude than the fluid viscosity μ.

Similarly we have for the (linearized) equation expressing conservation of particles:

$$\frac{\partial \phi'}{\partial t}-V\frac{\partial \phi'}{\partial z}+w_p\frac{\mathrm{d}\phi_1}{\mathrm{d}z}+\phi_1\boldsymbol{\nabla}\cdot\boldsymbol{u}_p=D_p\nabla^2\phi', \tag{3.4}$$

where w_p is the upward vertical component of $\boldsymbol{u}_p$ and D_p is the particle diffusivity, again not simply a molecular transport but including transport due to the fluctuations in the particle velocity. Equation (3.4) is brought nearer to (2.3) if we replace $\rho_p\phi_0, \rho_p\phi_1, \rho_p\phi'$ by ρ_0, ρ_1, ρ' respectively; thus

$$\frac{\partial \rho'}{\partial t}-V\frac{\partial \rho'}{\partial z}+w_p\frac{\mathrm{d}\rho_1}{\mathrm{d}z}+\rho_1\boldsymbol{\nabla}\cdot\boldsymbol{u}_p=D_p\nabla^2\rho', \tag{3.5}$$

where, according to (3.1),
$$\frac{\mathrm{d}\rho_1}{\mathrm{d}z}=\rho_0 A\kappa\cos\kappa z. \tag{3.6}$$

It may now be seen that the equations (3.2), (3.3) and (3.5) governing the behaviour of a small disturbance to a steady sinusoidal distribution of particle concentration are broadly similar, although not identical, to the equations (2.1), (2.2) and (2.3) governing a disturbance to a steady sinusoidal distribution of density of stationary fluid. The differences are:

(i) the molecular transport coefficients μ and D appearing in (2.2) and (2.3) are replaced by the effective values μ_m and D_p appropriate to a particle dispersion;

(ii) the time derivative $\partial/\partial t$ appearing in (2.2) and (2.3) is replaced by

$$\frac{\partial}{\partial t}-V\frac{\partial}{\partial z} \tag{3.7}$$

in (3.3) and (3.5);

(iii) the fluid velocity $\boldsymbol{u}$ appearing in (2.1), (2.2) and (2.3) is replaced in some of the terms in (3.2), (3.3) and (3.5) by the mean mixture disturbance velocity $\boldsymbol{u}_m$ and in other terms by the mean particle disturbance velocity $\boldsymbol{u}_p$.

The change in the numerical values of the transport coefficients, (i) above, causes no problems.

Likewise the introduction of the convective rate of change, (ii) above, will be found to raise no mathematical problems which cannot be overcome. The sketch of the disturbance motion for a stationary stratified fluid in figure 6 in BN1 helps to make clear the consequences of a convective rate of change with the mean (vertical) particle velocity $-V$ when $\alpha/\kappa=0.1$. The particles are falling relative to the undisturbed concentration wave and carry both horizontal particle momentum and particle volume across the layers of positive and negative excess particle concentration. This transfer process obviously tends to suppress vertical variations of horizontal particle momentum and concentration, especially for the disturbances of large horizontal wavelength which are amplified by the tilting–sliding mechanism in the case $a \to 0$ (because for these large horizontal wavelengths the vertical gradients of the horizontal component of the disturbance velocity are much greater than those of the vertical component). I believe this vertical transfer process to be the primary effect of the presence of the discrete particles. It will be analysed mathematically in the next section.

The change (iii) however is awkward in that it introduces an additional dependent variable, namely $\boldsymbol{u}_p$, and in principle the set of equations (3.2), (3.3) and (3.5) must be supplemented by an additional (vector) equation, presumably the equation of motion of the particles alone or of the fluid alone, the mean velocities at a point being related by

$$\boldsymbol{u}_m = \phi \boldsymbol{u}_p + (1-\phi)\boldsymbol{u}_f. \tag{3.8}$$

If there is negligible slip between the two constituents as a consequence of the particle radius being very small, we have

$$V = 0, \quad \boldsymbol{u}_m = \boldsymbol{u}_p = \boldsymbol{u}_f, \tag{3.9}$$

and the equations (3.2), (3.3) and (3.5) are then identical in form to (2.1), (2.2) and (2.3). But if $a \neq 0$, then $V \neq 0$ and $\boldsymbol{u}_m$ is not necessarily equal to $\boldsymbol{u}_p$.

The hypothesis to be made here is that there is no slip in the disturbance motion, so that $\boldsymbol{u}_m = \boldsymbol{u}_p$, even when $a \neq 0$. This is valid for sufficiently small a, and we may estimate the implied restriction on the value of a by comparing the particle relaxation time with the time $\frac{1}{2}\lambda/V$ (where $\lambda = 2\pi/\kappa$) taken by a particle to fall with mean speed V through one of the concentration layers of thickness $\frac{1}{2}\lambda$ (the 'traverse time'). The relaxation time is defined as the e-folding time of the relative velocity of fluid and particles on which no forces other than fluid resistance are acting. The relaxation time is usually defined for an isolated particle of mass m, in which case it may be denoted by τ_0 and is equal to $m/6\pi a\mu$ (assuming low-Reynolds-number flow around the particle). Here we are concerned with a homogeneous dispersion of particles of (approximately) uniform mean concentration ϕ_0 with uniform mean particle velocity, and since the mean fall speed (relative to zero-volume-flux axes) of particles in such a dispersion due to the uniform gravitational force mg is $U(\phi_0)$, the 'collective' relaxation time is

$$\tau_c = \frac{U(\phi_0)}{g} \quad \left(= \frac{\tau_0 U(\phi_0)}{U(0)}\right). \tag{3.10}$$

Hence we have

$$\frac{\text{particle relaxation time } (\tau_c)}{\text{particle traverse time}} = \frac{2U(\phi_0)V}{\lambda g}$$

$$= \frac{2U_0^2}{\lambda g}\left|\frac{\phi U}{U_0}\frac{\mathrm{d}(U/U_0)}{\mathrm{d}\phi}\right|_{\phi=\phi_0}, \tag{3.11}$$

where U_0 is the speed of fall of an isolated particle.

The ratio (3.11), which must be small for our hypothesis to be justified, is the product of a Froude number based on half the vertical wavelength of the undisturbed mean density distribution and a numerical factor determined by the dependence of the mean fall speed in a homogeneous dispersion on concentration. The second of these factors in particular may be significantly smaller than unity for practical particle sizes and concentrations. For instance, for particles of density 1 gm cm^{-3} and radius 50 μm in air, the Froude number $2U_0^2/\lambda g$ is about unity when $\lambda = 2$ cm; and use of the Richardson–Zaki correlation, namely

$$U(\phi) = U_0(1-\phi)^p, \tag{3.12}$$

where p ranges between 2.5 and 5.5 depending on the flow Reynolds number and is taken as 4.0 here for illustration, shows that the numerical factor is about 0.05 when $\phi_0 = 0.40$. Bearing in mind that U_0^2 is proportional to a^4, it is evident that the ratio (3.11) may be quite small for particle radii of the order of 100 μm or less. In these circumstances the difference between the disturbance particle and fluid velocities is small, and the hypothesis that

$$\boldsymbol{u}_m = \boldsymbol{u}_p$$

has some justification. In the next section we consider its consequences.

4. The instability of a sinusoidally stratified particle dispersion

With this hypothesis the equations (3.2), (3.3) and (3.5) governing a disturbance to a particle dispersion with a steady sinusoidal variation of concentration now coincide with (2.1), (2.2) and (2.3), provided that $\boldsymbol{u}$, μ, D and $\partial/\partial t$ in these latter equations are replaced by $\boldsymbol{u}_m$, μ_m, D_p and the operator (3.7) respectively. Just as u, v, ρ' and p' may be eliminated from (2.1), (2.2) and (2.3) to give (2.5), so an equation in the single dependent variable w_m may be obtained from (3.2), (3.3) and (3.5). Then, on assuming a disturbance of the form

$$w_m = e^{\gamma t} W_m(z) \cos \alpha x, \tag{4.1}$$

we have, in place of (2.7),

$$\left(\frac{\gamma}{D_p}+\alpha^2-\frac{V}{D_p}\frac{d}{dz}-\frac{d^2}{dz^2}\right)\left(\frac{\gamma}{\nu_m}+\alpha^2-\frac{V}{\nu_m}\frac{d}{dz}-\frac{d^2}{dz^2}\right)\left(\alpha^2-\frac{d^2}{dz^2}\right) = \frac{g\alpha^2}{D_m \nu_m \rho_0}\frac{d\rho_1}{dz}W_m$$
$$= \alpha^2\kappa^4 R\cos\kappa z\, W_m \tag{4.2}$$

in view of (3.1), where $\nu_m = \mu_m/\rho_0$ and

$$R = \frac{gA}{\kappa^3 D_p \nu_m}. \tag{4.3}$$

We may now use the same mathematical procedure for the solving of equation (4.2) as was employed in BN1 on equation (2.7). It is assumed that the dependent variable W_m is a periodic function of z with period $2\pi/\kappa$ and so may be written as a Fourier series like (2.9) which is substituted in (4.2). On converting all terms in (4.2) to Fourier series and equating coefficients of like terms we obtain a three-term recurrence relation. As a consequence of the appearance of odd-order derivatives with respect to z in (4.2), the sine and cosine terms in the Fourier series are now not independent.

Proceeding on the assumption that the Fourier coefficients F_n and G_n decrease rapidly as n increases – which is suggested by inspection and can be confirmed *a posteriori* – we truncate the Fourier series at $n = 2$. The condition for non-zero Fourier coefficients to exist is then

$$\frac{L_0}{S^2} = \frac{L_2}{L_1 L_2 - S^2} + \frac{L_{-2}}{L_{-1}L_{-2}-S^2}, \tag{4.4}$$

where
$$L_n = \left(\frac{\gamma}{\kappa^2 D_p}+\frac{\alpha^2}{\kappa^2}-\frac{inV}{\kappa D_p}+n^2\right)\left(\frac{\gamma}{\kappa^2 \nu_m}+\frac{\alpha^2}{\kappa^2}-\frac{inV}{\kappa \nu_m}+n^2\right)\left(\frac{\alpha^2}{\kappa^2}+n^2\right) \tag{4.5}$$

and
$$\mathrm{i} = (-1)^{\frac{1}{2}}, \quad S = \frac{\alpha^2}{2\kappa^2}R. \tag{4.6}$$

The amplitude of the complex quantity L_n increases rapidly with n, and it appears that the smallest root of (4.4) for S for given γ (this being the root of greatest physical interest) is such that

$$S^2 \ll |L_1 L_2|, |L_{-1} L_{-2}|,$$

in which event the approximate solution of (4.4) is

$$S^2 = \frac{L_0 L_1 L_{-1}}{L_1 + L_{-1}}. \tag{4.7}$$

If $V = 0$, corresponding to the case of a continuous fluid of non-uniform density considered in BN1, L_n is real and $L_n = L_{-n}$ so (4.7) reduces to

$$S^2 = \tfrac{1}{2} L_0 L_1, \tag{4.8}$$

which is identical with the first of the two relations in (5.15) of BN1 as expected. (Note that as defined here L_n is κ^{-6} times the definition adopted in BN1.) If $V \neq 0$, (4.7) involves the additional dimensionless parameter

$$J = \frac{V}{\kappa (D_p \nu_m)^{\frac{1}{2}}}. \tag{4.9}$$

Equation (4.7) is a sixth-order algebraic equation for γ as a function of $R, \alpha/\kappa, P(=\nu_m/D_p)$ and J which needs to be solved numerically. However, the condition for a disturbance to be neutrally stable ($\gamma = 0$) can be found analytically to be

$$R^2_{\gamma=0} = \frac{2\beta(\beta-1)(\beta^2+PJ^2)(\beta^2+P^{-1}J^2)}{\beta^2 - J^2}, \tag{4.10}$$

where
$$\beta = (\alpha^2+\kappa^2)/\kappa^2.$$

Note again that when $J = 0$ this reduces to the first of the two relations (5.16) in BN1. Thus (4.10) shows the effect of the fall of particles through the concentration wave on the smallest value of R consistent with the disturbance being neutrally stable, for given α/κ.

Figure 1 shows R as a function of α/κ for various values of J for the case $P = 1, \gamma = 0$, according to (4.10). The tendency for the falling of the particles to suppress the growth of a disturbance is strongest at $\alpha/\kappa \ll 1$ and weakest at $\alpha/\kappa \gg 1$, as we expected. All the curves in figure 1 asymptote to the curve for $J = 0$ as $\alpha/\kappa \to \infty$, and become indistinguishable from it at a value of α/κ that increases with J. The curve for $J = 0$ itself has a simple asymptotic form (see (2.11)), and

$$R_{\gamma=0} \sim \surd 2(\alpha/\kappa)^4 \quad \text{for} \quad \alpha/\kappa \gg 1 \tag{4.11}$$

for any value of J. All the curves except those for $J < 1$ have a vertical tangent at a value of α/κ that decreases as J decreases to unity. We see from (4.10) that a necessary condition for a neutral disturbance to exist is $\beta > J$, that is,

$$\alpha^2/\kappa^2 > J - 1, \tag{4.12}$$

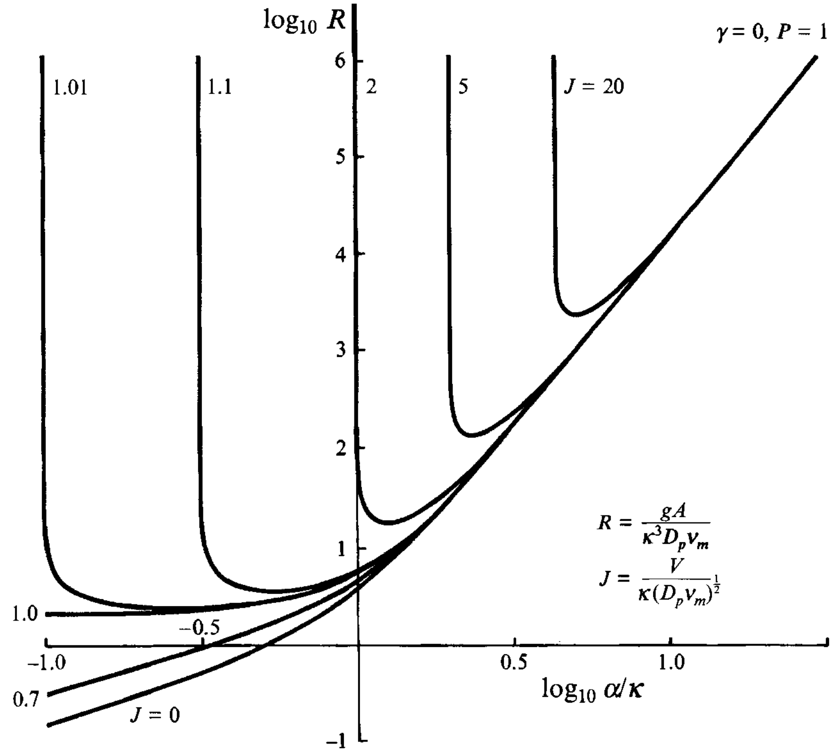

FIGURE 1. The Rayleigh number R as a function of horizontal wavenumber α for a neutral disturbance to a steady sinusoidally stratified particle dispersion for various values of the parameter J representing the effect of the fall of particles relative to the concentration wave; $P(= \nu_m/D_p) = 1$.

showing that when $J > 1$ there are values of α/κ with the upper bound $(J-1)^{\frac{1}{2}}$ for which the falling particles suppress the growth of a disturbance entirely.

The minimum value of $R_{\gamma=0}$ with respect to α/κ for given J remains at $\alpha/\kappa = 0$ when $J < 1$, and moves to a non-zero value of α/κ which increases as J increases above unity. Quantitatively, the value of α/κ at which $R_{\gamma=0}$ is a minimum for given $J(> 1)$ is roughly equal to the value of α/κ at which the appropriate curve in figure 1 develops a vertical tangent, that is,

$$R_{\gamma=0} \text{ is minimum at } \alpha/\kappa = (J-1)^{\frac{1}{2}}. \tag{4.13}$$

Moreover, the value of that minimum of $R_{\gamma=0}$ is roughly equal to the value given by the curve for $J = 0$ in figure 1 at $\alpha/\kappa = (J-1)^{\frac{1}{2}}$, that is, since the curve for $J = 0$ is given by (2.11),

$$(R_{\gamma=0})_{\min} \approx \surd 2 J^{\frac{3}{2}} (J-1)^{\frac{1}{2}}. \tag{4.14}$$

Similar approximate results for the conditions for maximum growth rate of a disturbance could be obtained.

The value $J = 1$ is thus critical, in the sense that when $J < 1$ a disturbance of sufficiently small horizontal wavenumber is neutrally stable, however small the Rayleigh number may be, just as in the case $J = 0$, but when $J > 1$ disturbances with small wavenumber are stable. We conclude the paper by estimating typical values of J in practice.

5. Typical numerical values of the parameter J

J has been defined in (4.9) as

$$J = V/\kappa(D_p \nu_m)^{\frac{1}{2}},$$

where (see (1.1))
$$V = \left| \phi \frac{dU}{d\phi} \right|_{\phi=\phi_0} \quad \text{and} \quad \nu_m = \frac{\mu_m}{\rho_p \phi_0}.$$

Here $U(\phi)$ is the mean fall velocity of particles (relative to zero-volume-flux axes) in a homogeneous dispersion with particle volume fraction ϕ. Accepted values of the transport coefficients D_p and μ_m associated with the random fluctuations of particle velocity are not available, either from theory or from observation, and the best we can do is to use a dimensional argument based on the simple postulate that the relevant properties of the dispersion are the particle radius a, the mean particle fall velocity $U(\phi_0)$, and the mean density of the dispersion $\rho_p \phi_0$. The kinematic coefficients D_p and ν_m are then functions of a and U of the form

$$D_p = \xi a U, \quad \nu_m = \eta a U, \tag{5.1}$$

where the numbers ξ and η are of order unity and may depend on ϕ_0. (Note that the molecular contributions to D_p and ν_m should be included in any consideration of what happens when $a \rightarrow 0$.) The parameter J thus becomes

$$J = \frac{1}{\kappa a (\xi \eta)^{\frac{1}{2}}} \left| \frac{\phi}{U} \frac{dU}{d\phi} \right|_{\phi=\phi_0},$$

and if we represent $U(\phi)$ by the Richardson–Zaki correlation (3.12) J may be written as

$$J = \frac{1}{\kappa a (\xi \eta)^{\frac{1}{2}}} \frac{p \phi_0}{1-\phi_0}, \tag{5.2}$$

where p varies from about 2.5 at high particle Reynolds number to about 5.5 at small Reynolds number. The primary instability of a gas-fluidized bed is likely to occur, as the bed is expanded, at fairly large particle volume fractions ϕ_0 around 0.4 or 0.5.

It thus appears that J is of the same order of magnitude as $(\kappa a)^{-1}$, where κ is the vertical wavenumber of the steady wavy disturbance generated by the primary instability of the uniform fluidized bed. Theory that assumes a specific mechanism for the suppression of a disturbance (Batchelor 1988) suggests values of $(\kappa a)^{-1}$ above unity for nearly neutral disturbances. Observational evidence of the value of $(\kappa a)^{-1}$ for gas-fluidized beds seems not to be available, perhaps because under common practical conditions the disturbances grow rapidly and quickly lose their periodic structure. The primary instability of a liquid-fluidized bed is weaker, and there is more direct evidence of the existence of growing plane wavy disturbances. Didwania & Homsy (1981) show photographs of plane waves propagating vertically and growing slowly in amplitude, and a representative value of $(\kappa a)^{-1}$ for these waves is about 40 (see their tables 1 and 4). It would be surprising if the value of $(\kappa a)^{-1}$ for a gas-fluidized bed (to which our two-phase flow analysis applies) were not also large compared with unity.

If, as is suggested by this discussion, the value of J exceeds unity, the estimates (4.13) and (4.14) of the critical conditions for the secondary instability are applicable. Disturbances grow only if the horizontal wavenumber α/κ exceeds $(J-1)^{\frac{1}{2}}$, and the value of α/κ at which $R_{\gamma=0}$ is minimum is roughly the same. The value of that minimum

Rayleigh number – an important quantity which is the main end product of our investigation – is given by (4.14), whence

$$(A_{\gamma=0})_{min} \approx \surd 2J^{\frac{3}{2}}(J-1)^{\frac{1}{2}}\frac{\kappa^3 D_p \nu_m}{g} \tag{5.3}$$

in view of the definition of R in (4.3). With use of the relations (5.1) this estimate of the minimum amplitude of the sinusoidal concentration wave for marginal secondary instability becomes

$$(A_{\gamma=0})_{min} = \surd 2J^{\frac{3}{2}}(J-1)^{\frac{1}{2}}\xi\eta(\kappa a)^3\frac{U^2}{ag}. \tag{5.4}$$

If now we (*a*) use the Richardson–Zaki correlation to replace $U(\phi)/U_0$ by $(1-\phi)^p$, (*b*) replace J by $(\kappa a)^{-1}$ times a factor of order unity (see above), and (*c*) ignore factors of order unity, we find

$$(A_{\gamma=0})_{\min} \sim (1-J^{-1})^{\frac{1}{2}}\kappa a\frac{U_0^2}{ag}(1-\phi_0)^{2p}. \tag{5.5}$$

Remember that this estimate is applicable only when $J > 1$, which is expected to be satisfied normally.

If the right-hand side of (5.5) is small compared with unity, the secondary instability always exists; if it is large compared with unity the secondary instability never exists; and if it is of order unity, there may be secondary instability at some value of $A(<1)$. The factors κa and $(1-\phi_0)^{2p}$ are appreciably smaller than unity and U_0^2/ag is appreciably larger, which leaves the magnitude of the product in (5.5) uncertain. Bearing in mind also the dropping of factors of order unity in the representation of J, D_p and ν_m it is impossible to say on *a priori* grounds whether the quantity (5.5) is larger or smaller than unity in a specific case. But since U_0^2 varies as a^4 for small particle Reynolds number and as a lesser positive power for large Reynolds number, it is evident that change of a with ϕ_0 fixed has a very strong influence on the value of the right-hand side of (5.5) and that almost any desired value can be realized by the appropriate choice of the value of a. In other words, the condition for secondary instability is satisfied for sufficiently small particles, provided that a remains sufficiently large for satisfaction of the condition for primary instability of the fluidized bed. It is intriguing that large values of a favour the primary instability whereas small values of a favour the secondary instability.

The horizontal wavenumber of the disturbance that is neutrally stable at the smallest value of the primary-wave amplitude A is given (see (4.13)) by

$$\frac{\alpha}{\kappa} \approx (J-1)^{\frac{1}{2}} \approx \left\{\frac{p\phi_0}{\kappa a(\xi\eta)^{\frac{1}{2}}(1-\phi_0)}-1\right\}^{\frac{1}{2}}$$

$$\approx (\kappa a)^{-\frac{1}{2}}\left\{\frac{p\phi_0}{(\xi\eta)^{\frac{1}{2}}(1-\phi_0)}\right\}^{\frac{1}{2}} \tag{5.6}$$

when $J \gg 1$. Thus the horizontal lengthscale is of the order of $(a/\kappa)^{\frac{1}{2}}$ which is smaller than the vertical lengthscale by the factor $(\kappa a)^{\frac{1}{2}}$. This horizontal lengthscale presumably determines the size of buoyant blobs (which become bubbles according to the suggestions made in Batchelor 1991) resulting from the growth and ultimate breakup of secondary disturbances. Note that the direction of the wavenumber α in the horizontal plane is arbitrary, and that the breakup of the growing secondary disturbance may therefore be expected to yield compact three-dimensional structures even though a single Fourier component with horizontal wavenumber yields a two-dimensional flow.

It is disappointing not to be able to make more definite quantitative predictions about the secondary instability of a fluidized bed. The problem here is, unusually, not the analysis of the two-phase flow but lies more in our ignorance of the values of parameters representing basic physical properties of a fluidized bed, such as the effective viscosity and particle diffusivity.

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